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Dipole and current fluctuations in the quantum one-component plasma at equilibrium

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Abstract. By an analysis of the current and dipole mean square fluctuations per unit volume for a quantum mechanical jellium in thermal equilibrium at temperature β^{-1}, we give a new proof of the sum rule $\int dx|x|^2 S(x) = \frac{3}{4}(\hbar\omega_p/\pi)\coth(\beta\hbar\omega_p/2)$ for the second moment of the charge-charge correlation function $S(x)$, where ω_p is the plasma frequency. No use is made of perturbation theories. The proof relies on correlation inequalities and sum rules which are valid in phases of charged systems having good screening properties.

Up to second order in Planck's constant, the current-current correlations have a non-integrable spatial decay even when the corresponding classical state shows Debye screening.

1. Introduction

In a recent work [1], correlation inequalities were used to determine the value of the bulk momentum fluctuations in a quantum mechanical system of density ρ in thermal equilibrium at temperature T. It was shown that the mean square momentum fluctuations by unit volume equal $m\rho k_B T$ (the classical equipartition law) whenever the potential is short ranged and the correlations have a sufficiently fast spatial decay (L^1-clustering).

In this paper, we extend this analysis to the quantum mechanical one component plasma (OCP) of particles with charge e and mass m in a neutralising background of density ρ (OCP). Quantities of interest in a charged system are the total dipole $D = \Sigma_i ex_i$ and current $J = \Sigma_i (e/m)p_i$, and their equilibrium fluctuations. These fluctuations can be calculated on the basis of the following heuristic remark: in the OCP, D and J are respectively proportional to the coordinate $X = \Sigma_i x_i$ and momentum $P = \Sigma_i p_i$ of the centre of mass of the whole system. Since the centre of mass decouples from the relative coordinates, it will only be subjected to the harmonic force $-m\omega_p^2 X$ ($\omega_p = (4\pi e^2\rho/m)^{1/2}$ = plasmon frequency) due to the charged background. We therefore expect that X and P will behave as the canonical variables of a macroscopic quantum oscillator of energy $(2m)^{-1}(P^2 + m^2\omega_p^2 X^2)$ and will be distributed accordingly in thermal equilibrium. Thus the equality of potential and kinetic energy for an oscillator leads to $\langle J^2\rangle = \omega_p^2\langle D^2\rangle$ and the mean square fluctuations of the dipole by unit volume are $|\Lambda|^{-1}\langle D^2\rangle = (1/8\pi)\hbar\omega_p\coth(\beta\hbar\omega_p/2)$.

The point of this paper is to give a precise version of this argument by means of correlation inequalities which characterise thermal states of infinitely extended systems. Clearly the result will not be true for a finite system, where boundary conditions have to be taken into account and the (x_i, p_i) do not obey the canonical commutation relations, but can only hold asymptotically in the thermodynamic limit. We take the same point of view as in [1]: we assume that we are given an infinitely extended system in terms of its reduced density matrices (RDM) (we do not prove the existence of the RDM for Coulomb systems, this can be done in some cases [2]), and the thermal equilibrium is characterised by the energy-entropy balance correlation inequalities [3, 4] (see appendix 1 for an elementary derivation of this inequality). With the help of appropriate cut-off functions, we define the local dipole and current of a finite region Λ of this infinite system. We then show that as $\Lambda \to \mathbb{R}^3$, these quantities behave indeed as the variables of a single quantum harmonic oscillator with frequency ω_p.

When we express the dipole fluctuations in terms of the structure function $S(x)$ of the OCP (the static charge-charge correlation function), the result is also formulated as an exact sum rule for the second moment of $S(x)$ (formula (4.1) of proposition 2). This sum rule can also be obtained in the framework of the linear response theory noting that, in the long wavelength limit, the imaginary part of the inverse dielectric function $\varepsilon^{-1}(k, \omega)$ (e.g. calculated in the RPA approximation) is peaked at the plasmon frequency, i.e. $\lim_{k\to 0} \operatorname{Im} \varepsilon^{-1}(k, \omega) = -\frac{1}{2}\pi(\delta(\omega - \omega_p) - \delta(\omega + \omega_p))$ [5, 6]. This, combined with the standard relation between $\varepsilon(k, \omega)$ and the structure function $S(x)$, leads to the sum rule of proposition 2†. Our results, obtained by a completely different method, confirm that the above behaviour of $\operatorname{Im} \varepsilon^{-1}(k, \omega)$ as $k \to 0$ is exact, a conclusion which is usually to be inferred from the fact that it saturates the frequency sum rules [5, 6].

As $\hbar \to 0$, (4.1) reduces to the familiar Stillinger-Lovett second moment condition expressing the perfect shielding in a classical plasma [7].

Our proofs rely on certain spatial cluster properties of the RDM which express the screening effects in charged systems. These properties are described in § 2 where the general formalism of infinite quantum states is briefly recalled. Although these properties have not yet been rigorously proven in the quantum case (see [8] for classical systems), we expect them to hold at least in the homogeneous phase at sufficiently high temperature. In particular, we assume that the truncated charge-charge correlations have sufficiently fast decay properties (integrability of second moments) and that the electrostatic charge and dipole sum rules, and an off-diagonal sum rule established in [9], are valid. However, as seen from the $\hbar$-expansion (appendix 2), the velocity-velocity correlations are not expected to be integrable in quantum Coulomb systems. Consequently, the current-current fluctuations can only be defined by conditionally convergent integrals. This implies that we have to make a special choice of the cut-off functions (associated here with cylindrical regions) and to remove the cut-offs by taking limits in the appropriate order.

The results for the OCP depend essentially on the fact that in this system the charge current is proportional to the mass current. This is no more true in multicomponent systems made of several species of masses m_α and charges e_α: the analysis cannot be generalised there in a straightforward manner. The equality of dipole and current fluctuations (up to a factor involving the individual plasmon frequencies) remains true (see § 3), but no simple expression has been found for the second moment of the charge-charge correlation function. It can however be checked that, using the methods of [1], the mean square momentum fluctuations of a multicomponent system (without background) satisfying suitable cluster properties are still given by the classical law $\sum_\alpha \rho_\alpha m_\alpha k_B T$.

† We are indebted to B Jancovici for pointing out to us this derivation of the formula (4.1).

In the last section, we consider a system of particles in a constant magnetic field in the 3-direction. By an analogous use of the correlation inequalities, the 1-2 components of the total velocity are shown to behave as the variables of a quantum oscillator of cyclotronic frequency $\omega_B = eBm^{-1}$ and to have the corresponding thermal averages.

2. General setting

2.1. The local observables and the state

We recall the local description of an infinitely extended quantum mechanical state. For simplicity, we consider one species of particles without internal degrees of freedom. The setting is the same as that given in § 2 of [1].

Let $\mathcal{H} = \mathcal{L}^2(\mathbb{R}^3)$ be the one particle space. The local observables of a particle consist of the $*$-algebra $\mathfrak{A}$ generated by the operators of the form $R(p)f(q)$ where R is a polynomial in the momentum p and f is a infinitely differentiable function of the position q, with compact support. Here $q^r = x^r$ and $p^r = -\mathrm{i}\hbar\, \partial/\partial x^r$ $(r = 1, 2, 3)$ are the usual Schrödinger representation of position and momentum on $\mathcal{L}^2(\mathbb{R}^3)$. A local observable of an n-particle system is an operator $A(1, \ldots, n)$ belonging to the symmetrised tensor product $\mathfrak{A}^{(n)} = (\mathfrak{A}^{\otimes n})_{\text{sym}}$. $A(1, \ldots, n)$ acts on $\mathcal{H}_\pm^{\otimes n}$, the properly symmetrised $(+)$ or antisymmetrised $(-)$ n-particle space. A local observable A of the infinite system is a sum $A = \Sigma_n A_n$ of n-body operators where only a finite number of A_n are different from zero. An n-body operator can be formally represented in second quantised form by

$$A_n = \frac{1}{n!} \int \mathrm{d}x_1 \ldots \mathrm{d}x_n\, \mathrm{d}y_1 \ldots \mathrm{d}y_n \langle x_1 \ldots x_n | A(1 \ldots n) | y_1 \ldots y_n \rangle$$
$$\times a^+(x_1) \ldots a^+(x_n) a(y_n) \ldots a(y_1) \tag{2.1}$$

where $\langle x_1 \ldots x_n | A(1 \ldots n) | y_1 \ldots y_n \rangle$ is the kernel of $A(1 \ldots n)$ in the configuration representation and the $a^+(x)$, $a(y)$ satisfy the canonical commutation relations

$$[a(x), a^+(y)]_\mp = \delta(x - y), \qquad [a(x), a(y)]_\mp = 0. \tag{2.2}$$

As a result of the Wick ordering of the creation and annihilation operators, a product AB can again be written as a sum of n-body operators of the form (2.1). This defines the product rule in the algebra of local observables.

The state $\langle \cdots \rangle$ of the infinite system is given in terms of its reduced density matrices $\rho^{(n)}$ (positive operators on $\mathcal{H}_\pm^{\otimes n}$) with kernels formally written as

$$\langle y_1 \ldots y_n | \rho^{(n)} | x_1 \ldots x_n \rangle = \langle a^+(x_1) \ldots a^+(x_n) a(y_n) \ldots a(y_1) \rangle. \tag{2.3}$$

The average of a n-body operator is then (set $(x)_n = (x_1 \ldots x_n)$):

$$\langle A_n \rangle = \frac{1}{n!} \int \mathrm{d}(x)_n\, \mathrm{d}(y)_n \langle (x)_n | A(1 \ldots n) | (y)_n \rangle \langle (y)_n | \rho^{(n)} | (x)_n \rangle$$
$$= \frac{1}{n!} \int \mathrm{d}(x)_n \langle (x)_n | A(1 \ldots n) \rho^{(n)} | (x)_n \rangle. \tag{2.4}$$

When A belongs to $\mathfrak{A}^{(n)}$, the kernels $\langle (x)_n | A(1 \ldots n) | (y)_n \rangle$ have to be understood in the sense of distributions, and the functions $\langle (y)_n | \rho^{(n)} | (x)_n \rangle$ are assumed to be infinitely differentiable in x and y. Hence the average (2.4) is well defined for $A \in \mathfrak{A}^{(n)}$.

The $\langle (x)_n|\rho^{(n)}|(y)_n\rangle$ have the symmetry under the permutations of the arguments induced by the representation (2.3) and the statistics. Moreover, the state is translation invariant if

$$\langle x_1+z\ldots x_n+z|\rho^{(n)}|y_1+z\ldots y_n+z\rangle = \langle (x)_n|\rho^{(n)}|(y)_n\rangle \tag{2.5}$$

for all $z\in\mathbb{R}^3$ and rotation invariant if

$$\langle Rx_1\ldots Rx_n|\rho^{(n)}|Ry_1\ldots Ry_n\rangle = \langle (x)_n|\rho^{(n)}|(y)_n\rangle \tag{2.6}$$

for any rotation R of $\mathbb{R}^3$.

Finally the state is time-reversal invariant if $\langle (x)_n|\rho^{(n)}|(y)_n\rangle$ are real, or equivalently if

$$\langle \tau(A)\rangle = \langle A^+\rangle \qquad \forall \text{ local } A, \tag{2.7}$$

where τ is the time-reversal operation: $\tau(p)=-p$, $\tau(q)=q$.

2.2. *Equilibrium states of Coulomb systems*

The Hamiltonian of a one-component Coulomb system is formally defined by

$$H = T+U \tag{2.8}$$

$$T=\int \mathrm{d}x\,\mathrm{d}y\langle x||p|^2/2m|y\rangle a^+(x)a(y) \tag{2.9}$$

$$U=\tfrac{1}{2}\int \mathrm{d}x\,\mathrm{d}y\,:Q(x)\phi(x-y)Q(y): \tag{2.10}$$

where $Q(x)=e(a^+(x)a(x)-\rho)$ is the charge density, $-e\rho$ the density of a uniformly charged background (jellium system), and

$$\phi(x)=|x|^{-1}+\phi_s(x), \tag{2.11}$$

ϕ_s is a finite range potential. The dots mean Wick ordering.

The states that we shall consider are translation, rotation and time-reversal invariant. Moreover, they are locally neutral i.e.

$$\langle Q(x)\rangle = e(\langle x|\rho^{(1)}|x\rangle-\rho)=0 \tag{2.12}$$

and they are assumed to satisfy the following cluster property with respect to the charge

$$\langle Q(x)a^+(x_1)\ldots a^+(x_n)a(y_1)\ldots a(y_n)\rangle = \mathrm{O}(|x|^{-\eta}) \tag{2.13}$$

for fixed $x_1\ldots x_n y_1\ldots y_n$ and some $\eta>3$. Other types of cluster properties are discussed in § 2.3.

An equilibrium state at inverse temperature β will be characterised as follows. It is stationary under time evolution

$$\langle [H,A]\rangle = 0 \tag{2.14}$$

and verifies the correlation inequalities [3, 4] (see appendix 1)

$$\beta\langle A^+[H,A]\rangle \geqslant \langle A^+A\rangle \ln\frac{\langle A^+A\rangle}{\langle AA^+\rangle} \tag{2.15}$$

for each local A.

Because of the long range of the Coulomb potential, $[H, A]$ is not local. However, for the states having the cluster property (2.13) the left-hand sides of (2.14) and (2.15) are well defined. It is easy to check that $[T, A]$ is local. Moreover, if A and B are two operators of the form (2.1), we write explicitly

$$\langle B[U, A_n]\rangle = \frac{1}{n!}\int d(x)_n\, d(y)_n \langle (x)_n|A(1\ldots n)|(y)_n\rangle$$
$$\times\Big(e^2\sum_{i<j}(\phi(x_i-x_j)-\phi(y_i-y_j))\langle Ba^+(x_1)\ldots a^+(x_n)a(y_n)\ldots a(y_1)\rangle$$
$$+e\sum_i\int dx(\phi(x_i-x)-\phi(y_i-x))\langle B{:}Q(x)a^+(x_1)\ldots a(y_1){:}\rangle\Big). \qquad (2.16)$$

The condition (2.13) implies that $\langle B{:}Q(x)a^+(x_1)\ldots a(y_1){:}\rangle = O(|x|^{-\eta})$ for fixed $x_1\ldots x_n y_1\ldots y_n$ and any local B, showing that the x integral occurring in the last term of the right-hand side of (2.16) is absolutely convergent. For any local B, by using Wick ordering, $\langle B{:}Q(x)a^+(x_1)\ldots a^+(x_n)a(y_n)\ldots a(y_1){:}\rangle$ can be expressed in terms of reduced density matrices as a sum of averages of the type (2.4).

When A and B are one-body operators, we find explicitly

$$\langle B[T, A]\rangle = \int dx_1\langle x_1|\rho^{(1)}B(1)[p_1^2/2m, A(1)]|x_1\rangle$$
$$+\int dx_1\, dx_2\langle x_1x_2|\rho^{(2)}B(1)[p_2^2/2m, A(2)]|x_1x_2\rangle \qquad (2.17a)$$

$$\langle B[U, A]\rangle = e^2\int dx_1\, dx_2\langle x_1x_2|\rho^{(2)}B(1)[\phi(12), A(2)]|x_1x_2\rangle$$
$$+e^2\int dx_1\, dx_2\langle x_1x_2|(\rho^{(2)}-\rho^{(1)}\rho^{(1)})B(1)[\phi(12), A(1)]|x_1x_2\rangle$$
$$+e^2\int dx_1\, dx_2\, dx_3\langle x_1x_2x_3|(\rho^{(3)}-\rho^{(2)}\rho^{(1)})B(1)[\phi(23), A(2)]|x_1x_2x_3\rangle \qquad (2.17b)$$

with $\phi(kl)=\phi(q_k-q_l)$. (2.16) and (2.17a, b) will be taken as definitions for the left-hand side of the inequality (2.15).

2.3. *Cluster properties and sum rules*

In the following we shall only consider homogeneous phases of the OCP having good screening properties. The RDM of such phases have cluster properties and obey sum rules typical for charged systems, which we now describe.

For notational convenience we introduce the density and momentum correlations

$$\langle\rho^{(n)}p_{i_1}^{r_1}\ldots p_{i_s}^{r_s}\rangle(x_1\ldots x_{n-1}) = \langle x_1\ldots x_{n-1}, 0|\rho^{(n)}p_{i_1}^{r_1}\ldots p_{i_s}^{r_s}|x_1\ldots x_{n-1}, 0\rangle$$
$$r_j=1,2,3, \qquad i_j=1,\ldots,n, \qquad s=0,1,2,\ldots$$

and distinguish between the density-density, momentum-density and momentum-momentum correlations.

The density-density correlations are assumed to have fast cluster properties as in the classical case. More specifically, we assume

$$\text{(i)} \qquad \int \mathrm{d}x |x|^2 |\langle \rho^{(2)} \rangle (x) - \rho^2| = \int \mathrm{d}x |x|^2 |\langle \rho_{\mathrm{T}}^{(2)} \rangle (x)| < \infty \tag{2.18}$$

$$\int \mathrm{d}x \int \mathrm{d}y |x| |\langle \rho_{\mathrm{T}}^{(3)} \rangle (x, y)| < \infty \tag{2.19}$$

where $\langle \rho_{\mathrm{T}}^{(3)} \rangle (x, y)$ is the fully truncated three-point function defined in the usual way.

We also assume that the momentum-density correlations are integrable†

$$\text{(ii)} \qquad \int \mathrm{d}x |\langle \rho^{(2)} p_1^r \rangle (x)| < \infty \tag{2.20}$$

$$\int \mathrm{d}x |\langle \rho^{(2)} p_1^r p_1^s \rangle (x) - \rho \langle \rho^{(1)} p_1^r p_1^s \rangle| < \infty. \tag{2.21}$$

The momentum-momentum correlations have, however, a slow clustering of the type

$$\text{(iii)} \qquad \langle \rho^{(2)} p_1^r p_2^s \rangle (x) = c(\partial^2 / \partial x^r \, \partial x^s)(|x|^{-1}) + \mathrm{O}(|x|^{-3-\varepsilon}) \qquad \varepsilon > 0. \tag{2.22}$$

These cluster properties, although not rigorously proven, are in agreement with the behaviour of the quantum $\hbar^2$ correction to the classical limit (see appendix 2).

Moreover, the RDM obey the following sum rules expressing the shielding of a particle in the OCP at equilibrium. They are the charge sum rules

$$\int \mathrm{d}x (\langle \rho^{(2)} \rangle (x) - \rho^2) + \rho = 0 \tag{2.23}$$

$$\int \mathrm{d}x (\langle \rho^{(3)} \rangle (x, y) - \rho \langle \rho^{(2)} \rangle (y)) + 2 \langle \rho^{(2)} \rangle (y) = 0 \tag{2.24}$$

the dipole sum rules

$$\begin{aligned} &\int \mathrm{d}x \, x (\langle \rho^{(2)} \rangle (x - y) - \rho^2) + y \rho = 0 \\ &\int \mathrm{d}x \, x (\langle \rho^{(3)} \rangle (x, y) - \rho \langle \rho^{(2)} \rangle (y)) + y \langle \rho^{(2)} \rangle (y) = 0 \end{aligned} \tag{2.25}$$

and

$$\int \mathrm{d}x (\langle \rho^{(2)} (p_1^r)^2 \rangle (x) - \rho \langle \rho^{(1)} (p_1^r)^2 \rangle) + \langle \rho^{(1)} (p_1^r)^2 \rangle = 0. \tag{2.26}$$

For states having suitable cluster properties, these sum rules can be shown to be a necessary consequence of the equilibrium equations obeyed by the RDM [9]. We shall assume here that (2.23)-(2.26) hold and refer to [9] for their derivation. Equations (2.23)-(2.25) are the exact analogue of the charge and dipole sum rules discussed in [10, 11] for classical charged systems, whereas (2.26) has no classical equivalent. Equation (2.26) is a particular case of the off-diagonal sum rule established in [9, equation (3.2)].

† Notice that because of the rotation invariance of the state one has $\langle \rho^{(1)} p_1^r \rangle = 0$, $\langle \rho^{(1)} p_1^r p_1^s \rangle = \delta_{rs} \langle \rho^{(1)} (p^r)^2 \rangle$.

3. Dipole and current fluctuations

In this section, we show that the mean square dipole and current fluctuations are proportional, the proportionality factor being the square of the plasmon frequency. This holds in any stationary state having the properties given in § 2.3.

With the help of cutoff functions f and g (f and g are infinitely differentiable functions with compact support), we first define the local current J_f^r and dipole D_g^r ($r = 1, 2, 3$) associated with bounded space regions. They are one-body observables defined respectively by $(e/2m)(p^r f(q) + f(q)p^r)$ and $eq^r g(q)$, and are formally written as

$$J_f^r = (e/m) \int \mathrm{d}x\, \mathrm{d}y \langle x|\tfrac{1}{2}(p^r f + f p^r)|y\rangle a^+(x) a(y) \tag{3.1}$$

$$D_g^r = \int \mathrm{d}x\, x^r g(x) Q(x). \tag{3.2}$$

We choose f and g as smooth characteristic functions of a cylinder Λ of radius R and length $2L$ along the r axis. Setting $x = (x^r, \bar{x}^r)$ where $\bar{x}^r$ is the part of the vector x orthogonal to the r direction, we define

$$f(x) = \varphi(x^r)\chi(\bar{x}^r) \tag{3.3}$$

$$\begin{aligned} g(x) &= \psi(x^r)\chi(\bar{x}^r) \\ \chi(\bar{x}) &= \begin{cases} 1 & |\bar{x}| \leq R \\ 0 & |\bar{x}| > R+1 \end{cases} \\ \varphi(s) &= \begin{cases} 1 & |s| \leq L \\ 0 & |s| > L+1 \end{cases} \\ \psi(s) &= h(s/L) \end{aligned} \tag{3.4}$$

where $h(s)$ has compact support and satisfies

$$\begin{aligned} &h(s) = h(-s), \qquad h(1) = 1 \\ &\tfrac{1}{2}\int \mathrm{d}s (sh'(s))^2 = \tfrac{1}{2}\int \mathrm{d}s[(sh(s))']^2 = 1. \end{aligned} \tag{3.5}$$

The prime denotes a derivative.

For the sake of conciseness we shall not indicate the R, L dependence of f and g, and we simply write $J_f^r = J_\Lambda^r$ and $D_g^r = D_\Lambda^r$. It will always be understood that J_Λ^r and D_Λ^r are the current and dipole associated with the cylindrical region Λ and the cutoff functions (3.3)-(3.4).

We notice that $\langle J_\Lambda^r \rangle = 0$ and $\langle D_\Lambda^r \rangle = 0$ by time-reversal and space reflexion invariance.

In the lemmas 1 and 2 below, we show that the mean square fluctuations by unit volume of J_Λ^r and D_Λ^r are well defined and express them in terms of the momentum-momentum and density-density correlations.

Lemma 1. Under the clustering assumptions (2.18) and (2.22)

$$\lim_{L\to\infty} \lim_{R\to\infty} |\Lambda|^{-1} \langle (J_\Lambda^r)^2 \rangle \equiv (\Delta J^r)^2$$

exists with

$$(\Delta J^r)^2 = (e/m)^2\left(\langle\rho^{(1)}(p^r)^2\rangle + \int \mathrm{d}x^r \int \mathrm{d}\bar{x}^r \langle\rho^{(2)} p_1^r p_2^r\rangle(x)\right). \tag{3.6}$$

Proof. The proof is similar to that of lemma 3.1 of [1], with some care due to the long-range velocity-velocity correlations. We first work out the average $\langle(J_\Lambda^r)^2\rangle$ by using Wick ordering to express it in terms of the reduced density matrices. Using also the time-reversal invariance as in [1] we get

$$(m/e)^2|\Lambda|^{-1}\langle(J_\Lambda^r)^2\rangle = \langle\rho^{(1)}p^{r^2}\rangle|\Lambda|^{-1}\int f^2(x)\,\mathrm{d}x \tag{3.7}$$

$$+|\Lambda|^{-1}\int \mathrm{d}x\langle\rho^{(2)}p_1^r p_2^r\rangle(x)\int \mathrm{d}y\, f(y)f(x+y) \tag{3.8}$$

$$+\tfrac{1}{4}\rho|\Lambda|^{-1}\int \mathrm{d}x(\nabla^r f(x))^2\,\mathrm{d}x$$

$$+\tfrac{1}{4}|\Lambda|^{-1}\int \mathrm{d}y\langle\rho_{\mathrm{T}}^{(2)}\rangle(y)\int \mathrm{d}x\,\nabla^r f(y+x)\nabla^r f(x). \tag{3.9}$$

With our choice (3.3) of f we clearly have

$$\lim_{L\to\infty}\lim_{R\to\infty}|\Lambda|^{-1}\int f^2(x)\,\mathrm{d}x = 1.$$

Notice that by the assumption (2.22),

$$\langle\rho^{(2)}p_1^r p_2^r\rangle(x^r,\bar{x}^r) = \mathrm{O}((x^{r^2}+|\bar{x}^r|^2)^{-3/2})$$

is integrable in $\bar{x}^r$ for fixed x^r. Moreover, since

$$\int \mathrm{d}\bar{x}^r \frac{\partial^2}{\partial x^{r^2}}|x|^{-1} = -4\pi\delta(x^r) = 0 \qquad \text{for } x^r \neq 0,$$

$\int \mathrm{d}\bar{x}^r\langle\rho^{(2)}p_1^r p_2^r\rangle(x^r,\bar{x}^r)$ is still integrable in x^r. Thus the contribution (3.8)

$$\int \mathrm{d}x^r(2L)^{-1}\int \mathrm{d}y^r\,\varphi(y^r+x^r)\varphi(y^r)$$

$$\times\left(\int \mathrm{d}\bar{x}^r\langle\rho^{(2)}p_1^r p_2^r\rangle(x^r,\bar{x}^r)(\pi R^2)^{-1}\int \mathrm{d}\bar{y}^r\,\chi(\bar{y}^r+\bar{x}^r)\chi(\bar{y}^r)\right)$$

tends to $\int \mathrm{d}x^r \int \mathrm{d}\bar{x}^r\langle\rho^{(2)}p_1^r p_2^r\rangle(x)$ by dominated convergence as we first let $R\to\infty$ and then $L\to\infty$. Finally the terms (3.9) involving $\nabla^r f(x) = (\nabla^r\varphi(x^r))\chi(\bar{x}^r)$ are $\mathrm{O}(L^{-1})$ uniformly in R and therefore vanish in the limit $L, R\to\infty$.

Introducing the charge-charge correlation function

$$\begin{aligned} S(x-y) &= \langle(Q(x)-\langle Q(x)\rangle)(Q(y)-\langle Q(y)\rangle)\rangle \\ &= e^2(\langle\rho^{(2)}\rangle(x-y)-\rho^2+\delta(x-y)\rho) \end{aligned} \tag{3.10}$$

we show that the dipole-dipole fluctuations are well defined as a consequence of the electroneutrality sum rule (2.23), which implies

$$\int \mathrm{d}x\, S(x) = 0, \qquad \int \mathrm{d}x\, xS(x) = 0. \tag{3.11}$$

The second relation follows from rotational invariance.

Lemma 2. Under the conditions (2.18) and (2.23)

$$\lim_{L\to\infty}\lim_{R\to\infty} |\Lambda|^{-1}\langle (D_\Lambda^r)^2\rangle \equiv (\Delta D^r)^2$$

exists with

$$(\Delta D^r)^2 = -\tfrac{1}{6}\int \mathrm{d}x |x|^2 S(x). \tag{3.12}$$

Proof. We have by the definition of $S(x)$

$$|\Lambda|^{-1}\langle (D_\Lambda^r)^2\rangle = \int \mathrm{d}x\, S(x)\Big((2L)^{-1}\int \mathrm{d}y^r (x^r+y^r)\psi(x^r+y^r)y^r\psi(y^r)\Big)$$
$$\times\Big((\pi R^2)^{-1}\int \mathrm{d}\bar{y}^r\,\chi(\bar{x}^r+\bar{y}^r)\chi(\bar{y}^r)\Big). \tag{3.13}$$

Since $S(x)$ has finite first and second moments, dominated convergence allows us to take the limit $R\to\infty$:

$$\lim_{R\to\infty}|\Lambda|^{-1}\langle (D_\Lambda^r)^2\rangle$$
$$=\int \mathrm{d}x\, S(x)\Big((2L)^{-1}\int \mathrm{d}s (x^r+s)\psi(x^r+s)s\psi(s)\Big)$$
$$=\int \mathrm{d}x\, S(x)\tfrac{1}{2}L^2\int \mathrm{d}s\,\gamma(s+x^rL^{-1})\gamma(s). \tag{3.14}$$

The scaling property of $\psi(s)$ has been used and $\gamma(s)=sh(s)$. Introducing the limited Taylor expansion $(0\leqslant\theta\leqslant 1)$

$$\gamma(s+x^rL^{-1}) = \gamma(s)+x^rL^{-1}\gamma'(s)+\tfrac{1}{2}(x^rL^{-1})^2\gamma''(s+\theta x^rL^{-1})$$

in (3.14) leads, with the sum rules (3.11), to

$$\lim_{L\to\infty}\lim_{R\to\infty}|\Lambda|^{-1}\langle (D_\Lambda^r)^2\rangle$$
$$=\lim_{L\to\infty}\tfrac{1}{2}\int \mathrm{d}x(x^r)^2S(x)\,\tfrac{1}{2}\int \mathrm{d}s\,\gamma''(s+\theta x^rL^{-1})\gamma(s)$$
$$=\tfrac{1}{2}\int \mathrm{d}s\,\gamma''(s)\gamma(s)\,\tfrac{1}{2}\int \mathrm{d}x(x^r)^2S(x)$$
$$=-\tfrac{1}{6}\int \mathrm{d}x|x|^2S(x)$$

where we have used (3.5)

$$\tfrac{1}{2}\int \mathrm{d}s\,\gamma''(s)\gamma(s) = -\tfrac{1}{2}\int \mathrm{d}s(\gamma'(s))^2 = -1.$$

The main result of this section is formulated in the following proposition.

Proposition 1. If the state has the properties given in § 2.3, one has

$$(\Delta J^r)^2 = \omega_p^2(\Delta D^r)^2 \tag{3.15}$$

with $\omega_p = (4\pi\rho e^2/m)^{1/2}$ = plasmon frequency.

Proof. The equality (3.15) follows from the identity

$$\langle J_\Lambda^r[H, D_\Lambda^r]\rangle + \langle [H, J_\Lambda^r]D_\Lambda^r\rangle = \langle [H, J_\Lambda^r D_\Lambda^r\rangle] = 0 \tag{3.16}$$

which is a consequence of the stationarity (2.14) of the state. We compute the terms of (3.16) in the limit $L, R \to \infty$ (the limit $R \to \infty$ being taken first). Details are given in appendix 3.

Writing $H = T + U$ (T = kinetic energy, U = potential energy), we see that the commutator

$$[H, D_\Lambda^r] = [T, D_\Lambda^r] = -\mathrm{i}\hbar e(2m)^{-1}\int \mathrm{d}x \int \mathrm{d}y \langle x|p\cdot\nabla(q^r g) + \nabla(q^r g)\cdot p|y\rangle a^+(x)a(y) \tag{3.17}$$

is a local approximation of the current. Thus, removing the cutoffs by taking the limit $R \to \infty$ first, we find, as in lemma 1, that the first term of the left-hand side of (3.16) gives

$$\lim_{L\to\infty}\lim_{R\to\infty} \mathrm{i}\hbar^{-1}|\Lambda|^{-1}\langle J_\Lambda^r[H, D_\Lambda^r]\rangle = (\Delta J^r)^2. \tag{3.18}$$

Working out the kinetic energy part of the second term in the left-hand side of (3.16) gives†

$$\lim_{L\to\infty}\lim_{R\to\infty} \mathrm{i}\hbar^{-1}|\Lambda|^{-1}\langle [T, J_\Lambda^r]D_\Lambda^r\rangle = -(e/m)^2\left(\langle \rho^{(1)}p_1^{r^2}\rangle + \int \mathrm{d}x\langle \rho_T^{(2)}p_1^{r^2}\rangle(x)\right). \tag{3.19}$$

This quantity vanishes as a consequence of the off-diagonal sum rule (2.26).

The potential energy part can be split into two contributions

$$\begin{aligned}\mathrm{i}\hbar^{-1}&|\Lambda|^{-1}\langle [U, J_\Lambda^r]D_\Lambda^r\rangle\\ &= \rho e^2 m^{-1}\int \mathrm{d}x\,|\Lambda|^{-1}\int \mathrm{d}z f(z)(z+x)^r g(z+x)\\ &\quad\times\nabla^r\int \mathrm{d}y\,\phi(x-y)S(y) &(3.20)\\ &\quad - e^4 m^{-1}|\Lambda|^{-1}\int \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\, r(x_1x_2x_3)(\nabla^r\phi)(x_1-x_2)f(x_1)x_2^r g(x_2) &(3.21)\end{aligned}$$

$r(x_1x_2x_3)$ involves the three-point function, and it is shown in appendix 3 that (3.21) vanishes as $L, R \to \infty$ because of the dipole sum rule (2.24).

† Notice that by time-reversal invariance $\langle D_\Lambda^r[H, J_\Lambda^r]\rangle = \langle [H, J_\Lambda^r]D_\Lambda^r\rangle$.

To find the limit of (3.20) we note that

$$\lim_{R\to\infty} \pi^{-1}R^{-2}\int \mathrm{d}\bar{z}^r\,\chi(\bar{z}^r)\chi(\bar{x}^r+\bar{z}^r)=1,$$

$$\begin{aligned}\lim_{R\to\infty} |\Lambda|^{-1}\int \mathrm{d}z f(z)(z+x)^r g(z+x)\\ &=(2L)^{-1}\int \mathrm{d}z^r\,\varphi(z^r)(x^r+z^r)\psi(x^r+z^r)\\ &=\tfrac{1}{2}L\int \mathrm{d}s\,\varphi(Ls)(s+x^rL^{-1})h(s+x^rL^{-1})\\ &=\tfrac{1}{2}L\int \mathrm{d}s\,\varphi(Ls)sh(s)+\tfrac{1}{2}x^r\int \mathrm{d}s\,\varphi(Ls)(sh(s))'(s+\theta x^rL^{-1})\\ &\qquad 0\leqslant\theta\leqslant 1,\end{aligned} \tag{3.22}$$

$$\lim_{L\to\infty}\tfrac{1}{2}\int \mathrm{d}s\,\varphi(Ls)(sh(s))'(s+\theta x^rL^{-1})=\tfrac{1}{2}\int_{-1}^{1}\mathrm{d}s(sh(s))'=h(1)=1. \tag{3.23}$$

The first term in the RHS of (3.22) being independent of x^r does not contribute to (3.20). We get therefore with (3.22) and (3.23)

$$\begin{aligned}\lim_{L\to\infty}\lim_{R\to\infty} \mathrm{i}\hbar^{-1}|\Lambda|^{-1}\langle[U,J_\Lambda^r]D_\Lambda^r\rangle\\ &=\rho e^2m^{-1}\int \mathrm{d}x\,x^r\Big(\nabla^r\int \mathrm{d}y\,\phi(x-y)S(y)\Big)\\ &=-\rho(6m)^{-1}e^2\int \mathrm{d}x|x|^2\nabla^2\int \mathrm{d}y\,\phi(x-y)S(y)\\ &=\frac{4\pi\rho e^2}{6m}\int \mathrm{d}x|x|^2S(x)\\ &=-\omega_\mathrm{p}^2(\Delta D^r)^2.\end{aligned} \tag{3.24}$$

The limit and integration by parts in (3.24) are justified by the fact that $S(y)$ carries no multipole moments and hence $\int \mathrm{d}y\,\phi(x-y)S(y)$ has a fast decay (see lemma 1 in [12]). Inserting (3.18) and (3.24) in (3.16) leads to the result of the proposition.

Proposition 1 can be generalised to multicomponent systems. If we have several species, say $\alpha=1,\ldots,N$, with Hamiltonian $H=T+U$,

$$T=\sum_{\alpha=1}^{N}\int \mathrm{d}x\,\mathrm{d}y\langle x|\frac{p^2}{2m_\alpha}|y\rangle a^+(\alpha,x)a(\alpha,y)$$

$$U=\tfrac{1}{2}\int \mathrm{d}x\,\mathrm{d}y\,:Q(x)V(x-y)Q(y):$$

$$Q(x)=\sum_{\alpha=1}^{N}e_\alpha a^+(\alpha,x)a(\alpha,x)+\rho_b$$

we define the local current of α particles and the local dipole by

$$J^r_{\alpha,\Lambda} = e_\alpha m_\alpha^{-1} \int \mathrm{d}x\, \mathrm{d}y \langle x|\tfrac{1}{2}(p^r f + f p^r)|y\rangle a^+(\alpha, x) a(\alpha, y)$$

$$D^r_\Lambda = \int \mathrm{d}x\, x^r g(x) Q(x)$$

with f and g as in (3.3), (3.4) and (3.5).

In a procedure completely analogous to the one followed in the OCP case, one can define current correlations

$$\langle J^r_{\alpha_1} J^r_{\alpha_2}\rangle = \lim_{L\to\infty} \lim_{R\to\infty} |\Lambda|^{-1} \langle J^r_{\alpha_1,\Lambda} J^r_{\alpha_2,\Lambda}\rangle$$

and express them in terms of RDM:

$$\langle J^r_{\alpha_1} J^r_{\alpha_2}\rangle = \frac{e_{\alpha_1}}{m_{\alpha_1}} \frac{e_{\alpha_2}}{m_{\alpha_2}} \left(\langle \rho^{(1)}_{\alpha_1} p^{r^2}\rangle \delta_{\alpha_1\alpha_2} + \int \mathrm{d}x^r \int \mathrm{d}\bar{x}^r \langle \rho_{\alpha_1\alpha_2} p^r_1 p^r_2\rangle(x) \right). \tag{3.25}$$

Dipole fluctuations are given by (3.12) provided S is defined as the multicomponent charge–charge correlation. Then, under good clustering hypothesis, the following identity holds

$$\begin{aligned} &\sum_{\alpha'} \langle J^r_\alpha J^r_{\alpha'}\rangle = (\omega^\alpha_{\mathrm{p}})^2 (\Delta D^r)^2 \\ &\omega^\alpha_{\mathrm{p}} = (4\pi e^2_\alpha \rho_\alpha / m_\alpha)^{1/2}. \end{aligned} \tag{3.26}$$

It is a consequence of stationarity $\langle [H, J^r_{\alpha,\Lambda} D^r_\Lambda]\rangle = 0$ and of the electrostatic sum rules for multicomponent systems [9].

4. Equilibrium dipole fluctuations in the OCP

In the preceding section, we proved that, in a Coulomb system, dipole and current fluctuations are equal up to dimensional factors, both being proportional to the second moment of the structure function. This identity was derived from the time-translation invariance of the state but no use has been made, so far, of the fact that the state is in thermodynamical equilibrium. We now make explicit use of the inequality (2.15) to determine the value of the dipole (and current) fluctuations of the OCP as a function of the temperature.

Proposition 2. If the state has the properties given in § 2.3, one has

$$(\Delta D^r)^2 = -\tfrac{1}{6} \int \mathrm{d}x |x|^2 S(x) = (4\pi)^{-1} \tfrac{1}{2} \hbar \omega_{\mathrm{p}} \coth(\tfrac{1}{2}\beta\hbar\omega_{\mathrm{p}}). \tag{4.1}$$

Proof. Since the idea of the proof is that in the OCP the total dipole and current behave as the canonical variables of a quantum harmonic oscillator of frequency ω_{p}, it is natural to make the choice

$$A_\Lambda = D^r_\Lambda + \mathrm{i}(\omega_{\mathrm{p}})^{-1} J^r_\Lambda \tag{4.2}$$

in the inequality (2.15). Writing (2.15) first for the pair A, A^+ and then exchanging the roles of A and A^+ gives

$$-\beta\frac{\langle A_\Lambda[H, A_\Lambda^+]\rangle}{\langle A_\Lambda A_\Lambda^+\rangle}\leq \ln\frac{\langle A_\Lambda^+ A_\Lambda\rangle}{\langle A_\Lambda A_\Lambda^+\rangle}\leq\beta\frac{\langle A_\Lambda^+[H, A_\Lambda]\rangle}{\langle A_\Lambda^+ A_\Lambda\rangle}. \tag{4.3}$$

We now calculate the various terms involved in (4.3) and remove the cutoffs, letting $R\to\infty$ and then $L\to\infty$.

We consider first

$$|\Lambda|^{-1}\langle A_\Lambda A_\Lambda^+\rangle = |\Lambda|^{-1}\langle D_\Lambda^{r2}\rangle + \omega_p^{-2}|\Lambda|^{-1}\langle J_\Lambda^{r2}\rangle + i\omega_p^{-1}|\Lambda|^{-1}\langle [J_\Lambda^r, D_\Lambda^r]\rangle \tag{4.4}$$

and as in (A3.2), we have

$$\lim_{L\to\infty}\lim_{R\to\infty} i|\Lambda|^{-1}\langle[J_\Lambda^r, D_\Lambda^r]\rangle$$
$$=\lim_{L\to\infty}\lim_{R\to\infty}\frac{\hbar e^2\rho}{m}|\Lambda|^{-1}\int dx f(x)\nabla^r(x^r g(x)) = \hbar e^2\rho m^{-1}. \tag{4.5}$$

Therefore we find from (4.5) and the result of proposition 1

$$\lim_{L\to\infty}\lim_{R\to\infty}|\Lambda|^{-1}\langle A_\Lambda A_\Lambda^+\rangle = 2(\Delta D^r)^2 + (4\pi)^{-1}\hbar\omega_p \tag{4.6}$$

and in the same way

$$\lim_{L\to\infty}\lim_{R\to\infty}|\Lambda|^{-1}\langle A_\Lambda^+ A_\Lambda\rangle = 2(\Delta D^r)^2 - (4\pi)^{-1}\hbar\omega_p. \tag{4.7}$$

Now, we calculate

$$|\Lambda|^{-1}\langle A_\Lambda^+[H, A_\Lambda]\rangle$$
$$= |\Lambda|^{-1}\langle D_\Lambda^r[H, D_\Lambda^r]\rangle + \omega_p^{-2}|\Lambda|^{-1}\langle J_\Lambda^r[H, J_\Lambda^r]\rangle$$
$$+ i\omega_p^{-1}|\Lambda|^{-1}(\langle D_\Lambda^r[H, J_\Lambda^r]\rangle - \langle J_\Lambda^r[H, D_\Lambda^r]\rangle). \tag{4.8}$$

It is sufficient to evaluate the first two terms of (4.8) since the last term has already been obtained in proposition 1 in the limit $L, R\to\infty$ with the result $-2\hbar\omega_p(\Delta D^r)^2$. (Notice that by time-reversal invariance $\langle D_\Lambda^r[H, J_\Lambda^r]\rangle = \langle[H, J_\Lambda^r]D_\Lambda^r\rangle$). Using time-reversal invariance, the first term of the right-hand side of (4.8) is

$$|\Lambda|^{-1}\langle D_\Lambda^r[H, D_\Lambda^r]\rangle = |\Lambda|^{-1}\langle D_\Lambda^r[T, D_\Lambda^r]\rangle = -|\Lambda|^{-1}\langle [T, D_\Lambda^r]D_\Lambda^r\rangle$$
$$= \tfrac{1}{2}|\Lambda|^{-1}\langle[D_\Lambda^r, [T, D_\Lambda^r]]\rangle$$
$$= \frac{\hbar^2 e^2}{2m}\rho|\Lambda|^{-1}\int dx|\nabla(x^r g(x))|^2. \tag{4.9}$$

Writing explicitly the integrand as

$$(\nabla^r(x^r\psi(x^r)))^2\chi^2(\bar{x}^r) + \sum_{s\neq r}(x^r\psi(x^r))^2(\nabla^s\chi(\bar{x}^r))^2$$

we see that the terms involving $\nabla^s\chi(\bar{x}^r)$ for $s\neq r$ are of the order of R^{-1} in (4.9) and do not contribute as $R\to\infty$ with L fixed. Therefore, since $\lim_{R\to\infty}(\pi R^2)^{-1}\int d\bar{x}^r\chi^2(\bar{x}^r)=1$,

$$\lim_{L\to\infty}\lim_{R\to\infty}|\Lambda|^{-1}\langle D_\Lambda^r[H,D_\Lambda^r]\rangle$$

$$=\hbar^2\frac{e^2\rho}{2m}\lim_{L\to\infty}(2L)^{-1}\int dx^r(\nabla^r(x^r\psi(x^r)))^2$$

$$=\hbar^2\frac{e^2\rho}{2m}\tfrac{1}{2}\int ds(sh(s))'^2=\hbar^2\omega_p^2/8\pi. \qquad (4.10)$$

It is shown in appendix 4 that the only non-vanishing contribution to $|\Lambda|^{-1}\langle J_\Lambda^r[H,J_\Lambda^r]\rangle$ is given by

$$\lim_{L\to\infty}\lim_{R\to\infty}\hbar^2\frac{e^4\rho^2}{2m^2}|\Lambda|^{-1}\int dx(\nabla^r\phi)(x)\int dy f(x-y)\nabla^r f(y)$$

$$=\lim_{L\to\infty}\lim_{R\to\infty}\frac{\hbar^2e^4\rho^2}{2m^2}(2L)^{-1}\int dx^r\int dy^r\varphi'(y^r)\varphi(x^r-y^r)$$

$$\times\left(\int d\bar{x}^r(\nabla^r\phi)(x^r,\bar{x}^r)(\pi R^2)^{-1}\int d\bar{y}^r\chi(\bar{y}^r)\chi(\bar{x}^r-\bar{y}^r)\right)$$

$$=\lim_{L\to\infty}\frac{\hbar^2e^4\rho^2}{2m^2}(2L)^{-1}\int dx^r\int dy^r\varphi'(y^r)\varphi(x^r-y^r)\int d\bar{x}^r(\nabla^r\phi)(x^r,\bar{x}^r)$$

$$=-\frac{\hbar^2\pi e^4\rho^2}{m^2}\lim_{L\to\infty}(2L)^{-1}\int ds\int dt\,\varphi'(s)\varphi(t-s)\,\mathrm{sgn}(t)$$

$$=\hbar^2\frac{2\pi e^4\rho^2}{m^2}\lim_{L\to\infty}(2L)^{-1}\int ds\,\varphi^2(s)$$

$$=\hbar^2\omega_p^4/8\pi. \qquad (4.11)$$

In obtaining (4.11) we have used

$$\int d\bar{x}\,\nabla^r((x^{r^2}+\bar{x}^2)^{-1/2})=-2\pi\,\mathrm{sgn}(x^r)$$

(the field due to an infinite plane of uniform density) and that the short-range part of the potential does not contribute in the limit $L\to\infty$. With (4.10) and (4.11) we find

$$\lim|\Lambda|^{-1}\langle A_\Lambda^+[H,A_\Lambda]\rangle=\hbar^2\omega_p^2(4\pi)^{-1}-2\hbar\omega_p(\Delta D^r)^2$$

$$=-\hbar\omega_p\lim|\Lambda|^{-1}\langle A_\Lambda^+A_\Lambda\rangle. \qquad (4.12)$$

By a similar calculation

$$\lim|\Lambda|^{-1}\langle A_\Lambda[H,A_\Lambda^+]\rangle=\hbar^2\omega_p^2(4\pi)^{-1}+2\hbar\omega_p(\Delta D^r)^2$$

$$=\hbar\omega_p\lim|\Lambda|^{-1}\langle A_\Lambda A_\Lambda^+\rangle. \qquad (4.13)$$

Inserting (4.6), (4.7), (4.12) and (4.13) in the inequality (4.3) gives

$$\ln\left(\frac{2(\Delta D^r)^2+\hbar\omega_p(4\pi)^{-1}}{2(\Delta D^r)^2-\hbar\omega_p(4\pi)^{-1}}\right)=\beta\hbar\omega_p$$

which is equivalent to (4.1).

5. Particles in a constant magnetic field

The Hamiltonian of a system of particles in a constant magnetic field B along the 3-axis is, in formal second quantised form,

$$H = T + U$$

$$T = \int \mathrm{d}x\, \mathrm{d}y\, \tfrac{1}{2}m\langle x|\, |v|^2\, |y\rangle a^+(x)a(y) \tag{5.1}$$

$$U = \tfrac{1}{2}\int \mathrm{d}x\, \mathrm{d}y\, :a^+(x)a(x)\phi(x-y)a^+(y)a(y): \tag{5.2}$$

where v^r is the velocity

$$v^1 = m^{-1}(p^1 + \tfrac{1}{2}eBq^2), \qquad v^2 = m^{-1}(p^2 - \tfrac{1}{2}eBq^1), \qquad v^3 = m^{-1}p^3 \tag{5.3}$$

$$[v^1, v^2] = \mathrm{i}\hbar m^{-1}\omega_B \qquad \omega_B = m^{-1}eB = \text{cyclotronic frequency.}$$

$\phi(x)$ is a spherically symmetric short-range potential (here the Coulomb potential is not taken into account: we assume $\int |\nabla\phi(x)|\, \mathrm{d}x < \infty$).

It is useful to note the following symmetries of this system. Let π^r, $r = 1, 2$, be the space reflexion operation of the r axis, i.e. $\pi^r(q^r) = -q^r$, $\pi^r(q^s) = q^s$ $s \neq r$, and τ the time reversal. Then the combined operation $\alpha^r = \tau\pi^r$ leaves the Hamiltonian invariant but changes the sign of velocities

$$\alpha^1(v^2) = -v^2, \qquad \alpha^2(v^1) = -v^1, \qquad \alpha^r(v^r) = v^r, \qquad r \neq 1, 2.$$

We assume that we have an equilibrium state of the infinite system which is translation invariant (on the algebra of gauge invariant observables) and has the symmetry $\langle\alpha^r(A)\rangle = \langle A^+\rangle$, $r = 1, 2$. Moreover, the density and velocity correlations

$$\langle\rho^{(n)}v_{i_1}^{r_1}\ldots v_{i_k}^{r_k}\rangle(x_1, \ldots, x_{n-1}) = (x_1 \ldots x_{n-1}, 0|\rho^{(n)}v_{i_1}^{r_1}\ldots v_{i_k}^{r_k}|x_1 \ldots x_{n-1}, 0\rangle$$

have integrable cluster properties.

Then, defining the local velocities

$$v_\Lambda^r = \int \mathrm{d}x\, \mathrm{d}y\, \tfrac{1}{2}\langle x|v^r f + f v^r|y\rangle a^+(x)a(y) \tag{5.4}$$

with f a smooth characteristic function of a sphere Λ of radius R, one proves

$$\lim_{R\to\infty} |\Lambda|^{-1}\langle(v_\Lambda^1)^2\rangle = \lim_{R\to\infty} |\Lambda|^{-1}\langle(v_\Lambda^2)^2\rangle \tag{5.5}$$

and

$$\lim_{R\to\infty} |\Lambda|^{-1}\langle(v_\Lambda^1)^2\rangle = \langle\rho^{(1)}v_1^1\rangle + \int \mathrm{d}x\langle\rho^{(2)}v_1^1v_2^1\rangle(x)$$

$$= (2m)^{-1}\hbar\omega_B\rho \coth(\tfrac{1}{2}\beta\hbar\omega_B). \tag{5.6}$$

The proof of (5.5) is analogous to that of proposition 1, starting from the identity $\langle[H, v_\Lambda^1 v_\Lambda^2]\rangle = 0$. Equation (5.6) is obtained from the inequality (2.15) as in proposition 2 with the choice $A_\Lambda = v_\Lambda^1 + \mathrm{i}v_\Lambda^2$. The details can be found in [13].

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Appendix 1

For the convenience of the reader, we give an elementary derivation of the inequality (2.15) for a finite quantum system with discrete spectrum. We refer to [4] for proofs of its equivalence with equilibrium conditions of infinite systems (KMS condition and thermodynamic stability).

If H has eigenvalues λ_i (repeated according to their multiplicities) with eigenfunctions φ_i and $\rho = z^{-1}\,\mathrm{e}^{-\beta H}$, $z = \mathrm{Tr}\,\mathrm{e}^{-\beta H}$, we have

$$\begin{aligned}
\mathrm{Tr}\,A^{+}[H, A]\rho &= z^{-1}\sum_{ij}\exp(-\beta\lambda_i)(\lambda_j-\lambda_i)|(\varphi_j, A\varphi_i)|^2 \\
&= -\beta^{-1}z^{-1}\sum_{ij}(\exp(-\beta\lambda_i)|(\varphi_j, A\varphi_i)|^2)\ln\exp[-\beta(\lambda_j-\lambda_i)] \\
&\geqslant -\beta^{-1}z^{-1}\Big(\sum_{ij}\exp(-\beta\lambda_i)|(\varphi_j, A\varphi_i)|^2\Big) \\
&\quad\times\ln\left(\frac{\Sigma_{ij}\exp(-\beta\lambda_i)|(\varphi_j, A\varphi_i)|^2\exp[-\beta(\lambda_j-\lambda_i)]}{\Sigma_{ij}\exp(-\beta\lambda_i)|(\varphi_j, A\varphi_i)|^2}\right) \\
&= \beta^{-1}(\mathrm{Tr}\,A^{+}A\rho)\ln(\mathrm{Tr}\,A^{+}A\rho/\mathrm{Tr}\,AA^{+}\rho).
\end{aligned}$$

The inequality follows from the convexity of the logarithm: for any sets of positive numbers $\{x_k\}$ and $\{a_k\}$,

$$\frac{\Sigma_k a_k \ln x_k}{\Sigma_k a_k} \leqslant \ln(\Sigma_k a_k x_k/\Sigma_k a_k).$$

Appendix 2

In this appendix, we check that, up to first order in $\hbar^2$, the momentum-density and momentum-momentum correlations satisfy properties (2.21), (2.22) and (2.26). More specifically, we shall use the Wigner-Kirkwood expansion to show that, if an equilibrium state of the OCP has good clustering and screening properties in its classical description, then the quantum correlations satisfy the conditions claimed in § 2.3.

The Wigner-Kirkwood expansion provides the quantum expectation value $\langle A\rangle_{\mathrm{Q}}$ of a local observable A in terms of purely classical averages [14]:

$$\langle A\rangle_{\mathrm{Q}} = \langle A\rangle + \hbar^2(\langle A\chi\rangle - \langle A\rangle\langle\chi\rangle) + \mathrm{o}(\hbar^2). \tag{A2.1}$$

Here, $\langle\ldots\rangle = \lim_{N\to\infty}\langle\ldots\rangle_N$ and $\langle\ldots\rangle_N$ is the canonical expectation defined by the Boltzmann factor

$$\exp\left[-\beta\left(\sum_{i=1}^{N}\frac{|p_i|^2}{2m} + U(q_1, \ldots, q_N)\right)\right].$$

The correction χ is given by

$$\chi=\sum_{k,l=1}^{N}\sum_{r,s=1}^{3}p_k^r p_l^s\left(\frac{\beta^3}{6m^2}\frac{\partial}{\partial q_k^r}\frac{\partial}{\partial q_l^s}U-\frac{\beta^4}{8m^2}\frac{\partial}{\partial q_k^r}U\frac{\partial}{\partial q_l^s}U\right)$$
$$+\sum_{k=1}^{N}\sum_{r=1}^{3}\left(-\frac{\beta^2}{4m}\frac{\partial^2}{\partial q_k^{r2}}U+\frac{\beta^3}{6m}\left(\frac{\partial}{\partial q_k^r}U\right)^2\right). \qquad \text{(A2.2)}$$

In the thermodynamic limit, the state $\langle\ldots\rangle$ is described by the corresponding classical correlation functions $\rho^{(n)}(\ldots)$. Those functions are known to satisfy the following sum rules [10]:

$$\int \mathrm{d}x\,\rho_{\mathrm{T}}^{(2)}(x,0)+\rho=0 \qquad \text{(A2.3)}$$

$$\int \mathrm{d}y\,\rho_{\mathrm{T}}^{(3)}(x,y,z)+2\rho\rho_{\mathrm{T}}(x,z)=0. \qquad \text{(A2.4)}$$

The following identities will be useful: the first one is obtained by explicit computation in the Maxwell distribution (α stands for a double index $\binom{r}{i}$)

$$\langle p_{\alpha_1}p_{\alpha_2}p_{\alpha_3}p_{\alpha_4}\rangle_{\text{Maxw.}}$$
$$=m^2\beta^{-2}\begin{cases}3 & \text{if } \alpha_1=\alpha_2=\alpha_3=\alpha_4\\ 1 & \text{if } (\alpha_1\alpha_2\alpha_3\alpha_4) \text{ has two pairs of identical indices}\\ 0 & \text{otherwise.}\end{cases} \qquad \text{(A2.5)}$$

The others follow from an integration by parts:

$$\frac{\partial^2}{\partial x^r\,\partial y^s}\rho^{(2)}(x,y)=\left\langle\sum_{i\neq j}\delta(x-q_i)\delta(y-q_j)\left(-\beta\frac{\partial}{\partial q_i^r}\frac{\partial}{\partial q_j^s}U+\beta^2\frac{\partial}{\partial q_i^r}U\frac{\partial}{\partial q_j^s}U\right)\right\rangle \qquad \text{(A2.6a)}$$

$$\frac{\partial^2}{\partial x^r\,\partial x^s}\rho^{(2)}(x,y)=\left\langle\sum_{i\neq j}\delta(x-q_i)\delta(y-q_j)\left(-\beta\frac{\partial}{\partial q_i^r}\frac{\partial}{\partial q_i^s}U+\beta^2\frac{\partial}{\partial q_i^r}U\frac{\partial}{\partial q_i^s}U\right)\right\rangle. \qquad \text{(A2.6b)}$$

Notice that for a translation invariant state in the thermodynamic limit, these expressions are identical (up to a sign) since then

$$\frac{\partial^2}{\partial x^r\,\partial y^s}\rho^{(2)}(x-y,0)+\frac{\partial^2}{\partial x^r\,\partial x^s}\rho^{(2)}(x-y,0)=0.$$

Let us show (2.22). By definition, $\langle\rho^{(2)}p_1^r p_2^s\rangle(x-y)$ is the expectation of the following observable:

$$A(x,y)=\sum_{i\neq j}p_i^r\delta(x-q_i)p_j^s\delta(y-q_j).$$

The classical mean is zero and the quantum correction is, using (A2.5) and (A2.6a),

$$\langle A(xy)\chi\rangle=\left\langle\sum_{i\neq j}\delta(x-q_i)\delta(y-q_j)\left(\frac{\beta}{3}\frac{\partial}{\partial q_i^r}\frac{\partial}{\partial q_j^s}U-\frac{\beta^2}{4}\frac{\partial}{\partial q_i^r}U\frac{\partial}{\partial q_j^s}U\right)\right\rangle$$
$$=\tfrac{1}{4}\nabla^r\nabla^s\rho^{(2)}(x,y)+\tfrac{1}{12}\beta\left\langle\sum_{i\neq j}\delta(x-q_i)\delta(y-q_j)\frac{\partial^2}{\partial q_i^r\,\partial q_j^s}U\right\rangle.$$

For the OCP, the potential energy is given by (2.10) provided that we take the classical charge density $Q(x)=e(\Sigma_i\,\delta(x-q_i)-\rho)$. Therefore, up to $o(\hbar^2)$ terms, we get

$$\langle\rho^{(2)}p_1^r p_2^s\rangle(x-y)$$
$$=\hbar^2\langle A(x,y)\chi\rangle=\tfrac{1}{4}\hbar^2\nabla^r\nabla^s\rho^{(2)}(x,y)-\tfrac{1}{12}\hbar^2\beta\rho^{(2)}(x,y)\nabla^r\nabla^s\phi(x-y). \qquad \text{(A2.7)}$$

Since $\rho_{\mathrm{T}}^{(2)}(x,y)$ clusters fast, the dominant contribution in the asymptotic limit is

$$\langle\rho^{(2)}p_1^r p_2^s\rangle(x)\simeq-\tfrac{1}{12}\hbar^2\beta\rho^2\nabla^r\nabla^s\phi(x) \qquad \text{as } |x|\to\infty. \qquad \text{(A2.8)}$$

Let us now turn to $\langle\rho^{(2)}p_1^r p_1^s\rangle(x-y)$, which is the average of the following expression:

$$A(x,y)=\sum_{i\neq j}p_i^r p_i^s\delta(x-q_i)\delta(y-q_j).$$

Applying (A2.5) we have

$$\langle A(x,y)\rangle=\delta_{r,s}m\beta^{-1}\rho^{(2)}(x,y) \qquad \text{(A2.9)}$$

$$\langle A(x,y)\chi\rangle=\delta_{r,s}\left\langle\sum_{i\neq j}\sum_k\sum_{t=1}^{3}\delta(x-q_i)\delta(y-q_j)\left(-\frac{\beta}{12}\frac{\partial^2}{\partial q_k^{t}}U+\frac{\beta^2}{24}\left(\frac{\partial}{\partial q_k^t}U\right)^2\right)\right\rangle \qquad \text{(A2.10)}$$

$$+\left\langle\sum_{i\neq j}\delta(x-q_i)\delta(y-q_j)\left(\frac{\beta}{3}\frac{\partial^2}{\partial q_i^r\,\partial q_i^s}U-\tfrac{1}{4}\beta^2\frac{\partial}{\partial q_i^r}U\frac{\partial}{\partial q_i^s}U\right)\right\rangle. \qquad \text{(A2.11)}$$

Up to a factor $m\beta^{-1}$, (A2.10) corresponds to the $\hbar^2$ correction of the two-point distribution function $\rho_{\mathrm{Q}}^{(2)}$. For an OCP, this function takes the following form [15]

$$\rho_{\mathrm{Q}}^{(2)}(x,y)=\rho^{(2)}(x,y)+(\hbar^2\beta/12m)\nabla^2\rho^{(2)}(x,y).$$

Using (A2.6*b*) to treat the second term (A2.11) and inserting all that into (A2.1) gives

$$\langle\rho^{(2)}p_1^r p_1^s\rangle(x-y)=m\beta^{-1}\rho_{\mathrm{Q}}^{(2)}(x,y)\delta_{r,s}-\tfrac{1}{4}\hbar^2\nabla^r\nabla^s\rho^{(2)}(x,y)$$
$$+\tfrac{1}{12}\hbar^2\beta\left\langle\sum_{i\neq j}\delta(x-q_i)\delta(y-q_j)\frac{\partial^2}{\partial q_i^r\,\partial q_i^s}U\right\rangle. \qquad \text{(A2.12)}$$

In order to handle the local singularity of the second derivative of the Coulomb potential, it is convenient to split the potential into a finite-range part ϕ_{f} plus a continuous long-range part ϕ_l:

$$\phi(x)=\phi_l(x)+\phi_{\mathrm{f}}(x).$$

The correlations $\rho^{(n)}(x,y\ldots)$ vanish at coincident points so that a product such as $\nabla^r\nabla^s\phi_{\mathrm{f}}(x-y)\rho^{(n)}(x,y\ldots)$ is indeed integrable with respect to x.

Making the potential energy explicit in (A2.12), we obtain

$$\langle\rho^{(2)}p_1^r p_1^s\rangle(x-y)$$
$$=\delta_{r,s}m\beta^{-1}\rho^{(2)}(x,y)-(1-\delta_{r,s})\tfrac{1}{4}\hbar^2\nabla^r\nabla^s\rho^{(2)}(x,y)$$
$$+\tfrac{1}{12}\hbar^2\beta e^2\left(\int dz(\nabla^r\nabla^s\phi_{\mathrm{f}})(x-z)\rho^{(3)}(x,y,z)+\nabla^r\nabla^s\phi_{\mathrm{f}}(x-y)\rho^{(2)}(x,y)\right)$$
$$+\tfrac{1}{12}\hbar^2\beta e^2\int dz(\nabla^r\nabla^s\phi_l)(x-z)(\rho^{(3)}(x,y,z)+(\delta(z-y)-\rho)\rho^{(2)}(x,y)).$$
$$\text{(A2.13)}$$

Along the same lines, one can evaluate

$$\langle\rho^{(1)}p_1^rp_1^s\rangle=\left\langle\sum_i p_i^rp_i^s\delta(x-q_i)\right\rangle$$

$$=\delta_{r,s}m\rho\beta^{-1}(1+\beta^2\hbar^2(12m)^{-1}\langle(\partial^2/\partial q_i^{r^2})U\rangle)$$

$$=m\rho\beta^{-1}\delta_{r,s}+\tfrac{1}{12}\beta e^2\hbar^2\int \mathrm{d}z(\nabla^r\nabla^s\phi_f)(x-z)\rho^{(2)}(x,z)$$

$$+\tfrac{1}{12}\beta e^2\hbar^2\int \mathrm{d}z(\nabla^r\nabla^s\phi_l)(x-z)\rho_T^{(2)}(x,z). \tag{A2.14}$$

We combine (A2.13) and (A2.14) to obtain truncated correlations:

$$\langle\rho^{(2)}p_1^rp_1^s\rangle(x,y)-\langle\rho^{(1)}p_1^rp_1^s\rangle\rho$$

$$=\delta_{r,s}m\beta^{-1}\rho_T^{(2)}(x,y) \tag{A2.15}$$

$$+(\delta_{r,s}-1)\tfrac{1}{4}\hbar^2\nabla^r\nabla^s\rho^{(2)}(x,y) \tag{A2.16}$$

$$+\tfrac{1}{12}\beta\hbar^2e^2\left(\int \mathrm{d}z\ \nabla^r\nabla^s\phi_f(x-z)(\rho^{(3)}(x,y,z)-\rho\rho^{(2)}(x,z))\right.$$

$$\left.+\nabla^r\nabla^s\phi_f(x-y)\rho^{(2)}(x,y)\right) \tag{A2.17}$$

$$+\tfrac{1}{12}\beta\hbar^2e^2\int \mathrm{d}z\ \nabla^r\nabla^s\phi_l(x-z)R(xyz) \tag{A2.18}$$

with

$$R(xyz)=\rho^{(3)}(xyz)-\rho\rho_T^{(2)}(xz)-\rho\rho^{(2)}(xy)+\delta(z-y)\rho^{(2)}(xy) \tag{A2.19}$$

$$=\rho_T^{(3)}(xyz)+S(y-z)+\delta(y-z)\rho_T^{(2)}(xy). \tag{A2.20}$$

(A2.15), (A2.16) and (A2.17) are integrable with respect to y for fixed x since the classical functions are. In the last term (A2.18), the contribution of the three-point truncated function (see (A2.20)) is integrable by the clustering (2.19). Moreover, the potential created by $S(x)=\rho_T(x,0)+\delta(x)\rho$ is $o(|y|^{-4})$ because S carries no electrostatic multipoles of order 0, 1, 2. All this proves (2.21).

We end this appendix by showing that (2.26) follows from the (classical) sum rules. The classical part of (2.26) is proportional to (A2.3). The $\hbar^2$ order contribution to (2.26) is made of two terms (up to constant factors) coming from (A2.17), (A2.18) and (A2.14). They are

$$\int \mathrm{d}y\left(\int \mathrm{d}z\ \nabla^{r^2}\phi_f(x-z)[\rho^{(3)}(x,y,z)-\rho\rho^{(2)}(x,z)]+2\nabla^{r^2}\phi_f(x-y)\rho^{(2)}(x,y)\right),$$

which is zero by combination of (A2.3) and (A2.4), and

$$\int \mathrm{d}y\int \mathrm{d}z\ \nabla^{r^2}\phi_l(x-z)R(xyz)+\int \mathrm{d}z\ \nabla^{r^2}\phi_l(x-z)\rho_T^{(2)}(x,z)$$

$$=\int \mathrm{d}y\frac{\partial^2}{\partial y^{r^2}}\left(\int \mathrm{d}z\ \phi_l(x-y-z)S(z)\right)=0$$

where (A2.4) has been used again.

Appendix 3

Proof of (3.18)

$$i\hbar^{-1}|\Lambda|^{-1}\langle J_\Lambda^r[T, D_\Lambda^r]\rangle$$
$$= (e/m)^2\langle\rho^{(1)}p^{r^2}\rangle|\Lambda|^{-1}\int dx\, f(x)\nabla^r(x^r g(x)$$
$$+(e/m)^2\sum_s\int dx\langle\rho^{(2)}p_1^r p_2^s\rangle(x)|\Lambda|^{-1}\int dy\, f(x+y)\nabla^s(y^r g(y))+o(1) \tag{A3.1}$$

where o(1) involves terms which are bilinear in derivatives like

$$|\Lambda|^{-1}\int dx(\nabla^s f)(x)\nabla^s(x^r g(x)),$$

and vanish in the limit $L, R\to\infty$ (these terms are the same as in the formulae (14) and (16) of [1]).

Now

$$\lim_{L\to\infty}\lim_{R\to\infty}|\Lambda|^{-1}\int dx\, f(x)\nabla^r(x^r g(x)) = \lim_{L\to\infty}(2L)^{-1}\int dt\,\varphi(t)(t\psi(t))'$$
$$= \lim_{L\to\infty}\tfrac{1}{2}\int dt\,\varphi(Lt)(th(t))' = h(1) = 1 \tag{A3.2}$$

and since

$$\lim_{R\to\infty}(\pi R^2)^{-1}\int d\bar{y}^r\chi(\bar{x}^r+\bar{y}^r)\nabla^s\chi(\bar{y}^r) = 0, \qquad s\neq r$$

one has, for a fixed x,

$$\lim_{L\to\infty}\lim_{R\to\infty}|\Lambda|^{-1}\int dy\, f(x+y)\nabla^s(y^r g(y))$$
$$= \delta_{r,s}\lim_{L\to\infty}2^{-1}\int dt\,\varphi(Lt+x^r)(th(t))' = \delta_{r,s}h(1) = \delta_{r,s}. \tag{A3.3}$$

$\langle\rho^{(2)}p_1^r p_2^s\rangle(x^r, \bar{x}^r)$ being integrable in $\bar{x}^r$ for x^r fixed, one can first take the limit $R\to\infty$, and then let $L\to\infty$ in the second term of the RHS of (A3.1). This leads as in lemma 1 to

$$\lim_{L\to\infty}\lim_{R\to\infty} i\hbar^{-1}|\Lambda|^{-1}\langle J_\Lambda^r[H, D_\Lambda^r]\rangle = (e/m)^2\left(\langle\rho^{(1)}p^{r^2}\rangle + \int dx^r\int d\bar{x}^r\langle\rho^{(2)}p_1^r p_2^r\rangle(x)\right)$$

and hence to (3.18).

Proof of (3.19)

$$i\hbar^{-1}|\Lambda|^{-1}\langle[T, J_\Lambda^r]D_\Lambda^r\rangle$$
$$= -(e/m)^2\sum_s\langle\rho^{(1)}p^r p^s\rangle|\Lambda|^{-1}\int dy\, f(y)\nabla^s(y^r g(y))$$

$$-(e/m)^2\sum_s\int \mathrm{d}x\langle\rho_\mathrm{T}^{(2)}p_1^r p_2^s\rangle(x)|\Lambda|^{-1}\int \mathrm{d}y f(y+x)\nabla^s(y^r g(y))+\mathrm{o}(1).$$

With (A3.2) and (A3.3) this gives (3.19) by dominated convergence.

Proof of (3.20)-(3.21)

Working out the commutator and the average according to (2.17*b*) gives (with the abbreviated notation $\rho(1,2,\ldots,n)=\langle x_1\ldots x_n|\rho^{(n)}|x_1\ldots x_n\rangle$)

$$\mathrm{i}\hbar^{-1}|\Lambda|^{-1}\langle[U,J_\Lambda^r]D_\Lambda\rangle$$
$$=-\frac{e^4}{m}|\Lambda|^{-1}\int \mathrm{d}1\,\mathrm{d}2\,\mathrm{d}3[\rho(123)-\rho(12)\rho+\delta(23)\rho(12)](\nabla^r\phi)(13)$$
$$\times f(1)x_2^r g(2)-\frac{e^4}{m}|\Lambda|^{-1}\int \mathrm{d}1\,\mathrm{d}2\,\rho_\mathrm{T}(12)(\nabla^r\phi)(12)f(1)x_1^r g(1). \qquad (\mathrm{A}3.4)$$

The second term of (A3.4) vanishes since, by the antisymmetry of $\nabla^r\phi(12)$ and translation invariance

$$\int \mathrm{d}x_2\,\rho_\mathrm{T}(12)(\nabla^r\phi)(12)=0. \qquad (\mathrm{A}3.5)$$

Rearranging terms and using (A3.5) again gives

$$\mathrm{i}\hbar^{-1}|\Lambda|^{-1}\langle[U,J_\Lambda^r]D_\Lambda^r\rangle$$
$$=-\frac{e^4\rho}{m}|\Lambda|^{-1}\int \mathrm{d}1\,\mathrm{d}2\,\mathrm{d}3[\rho(23)-\rho^2+\delta(23)\rho](\nabla^r\phi)(13)f(1)x_2^r g(2) \qquad (\mathrm{A}3.6)$$
$$-\frac{e^4}{m}|\Lambda|^{-1}\int \mathrm{d}1\,\mathrm{d}2\,\mathrm{d}3\,r(123)(\nabla^r\phi)(13)f(1)x_2^r g(2) \qquad (\mathrm{A}3.7)$$

where

$$\begin{aligned} r(123)&=\rho_\mathrm{T}(123)+\delta(23)\rho_\mathrm{T}(12)\\ \rho_\mathrm{T}(123)&=\rho(123)-\rho\rho_\mathrm{T}(23)-\rho\rho_\mathrm{T}(13)-\rho\rho(12). \end{aligned} \qquad (\mathrm{A}3.8)$$

With the definition (3.10) and a change of variables, (A3.6) is identical to (3.20).

The term (A3.7) is

$$\frac{e^4}{m}\int \mathrm{d}x\,\mathrm{d}y\,r(0xy)(\nabla^r\phi)(y)|\Lambda|^{-1}\int \mathrm{d}z f(z)(x^r+z^r)g(x+z).$$

By (3.22) and (3.23) and dominated convergence (see (2.19)) it tends to $-e^4m^{-1}\int \mathrm{d}y(\nabla^r\phi)(y)\int \mathrm{d}x\,x^r r(0xy)$ as $L, R\to\infty$. (Notice that $\int \mathrm{d}x\,\mathrm{d}y\,r(0xy)(\nabla^r\phi)(y)=0$ by the antisymmetry of $(\nabla^r\phi)(y)$).

Finally one checks easily from (A3.8) that the sum rules (2.25) imply $\int \mathrm{d}x\,x^r r(0xy)=0$, showing that (A3.7) does not contribute in the limit.

Appendix 4

We consider first the terms coming from the kinetic energy part of H, $|\Lambda|^{-1}\langle J_\Lambda^r[T,J_\Lambda^r]\rangle$. It has one- and two-body contributions. The one-body terms vanish in the limit

$L, R \to \infty$ for the same reason as in theorem 3.2 of [1] part (a): they all have a factor $|\Lambda|^{-1} \int dx\, K(x)$ where $K(x)$ involves derivatives of $f(x)$. The two-body terms vanish because of time reversal invariance.

The terms coming from the potential energy part of H have two- and three-body contributions which are computed according to (2.17b). The two-body part of $|\Lambda|^{-1}\langle J_\Lambda^r[U, J_\Lambda^r]\rangle$ is

$$i\hbar(e^2/m)^2|\Lambda|^{-1} \int dx \langle \rho^{(2)} p_1\rangle(x)(\nabla^r \phi)(x) \int dy (f(y) - f(x-y)) f(y) \tag{A4.1}$$

$$-\frac{\hbar^2 e^4}{2m^2}|\Lambda|^{-1} \int dx (\langle \rho^{(2)}\rangle(x) - \rho^2)(\nabla^r \phi)(x) \int dy (f(y) - f(x-y)) \nabla^r f)(y) \tag{A4.2}$$

$$+\frac{\hbar^2 e^4 \rho^2}{2m^2}|\Lambda|^{-1} \int dx (\nabla^r \phi)(x) \int dy (\nabla^r f)(y) f(x-y). \tag{A4.3}$$

Since $|\Lambda|^{-1} \int dy |f(y) - f(x-y)| \leq M$ and $\lim_{L\to\infty} \lim_{R\to\infty} |\Lambda|^{-1} \int (f(y) - f(x-y))\, dy = 0$ for fixed x, the terms (A4.1) and (A4.2) vanish in the limit by dominated convergence, and only (A4.3) contributes.

The three-body part of $|\Lambda|^{-1}\langle J_\Lambda^r[U, J_\Lambda^r]\rangle$ is identically zero because of time-reversal invariance as in theorem 3.2 of [1] part (a).

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